

# UNRAMIFIED BRAUER GROUPS OF FINITE AND INFINITE GROUPS

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*In loving memory of my father*

**ABSTRACT.** The Bogomolov multiplier is a group theoretical invariant isomorphic to the unramified Brauer group of a given quotient space. We derive a homological version of the Bogomolov multiplier, prove a Hopf-type formula, find a five term exact sequence corresponding to this invariant, and describe the role of the Bogomolov multiplier in the theory of central extensions. A new description of the Bogomolov multiplier of a nilpotent group of class two is obtained. We define the Bogomolov multiplier within K-theory and show that proving its triviality is equivalent to solving a long-standing problem posed by Bass. An algorithm for computing the Bogomolov multiplier is developed.

## 1. INTRODUCTION

In this paper we develop a homological version of a group theoretical invariant that has served as one of the main tools in studying the problem of stable rationality of quotient spaces. Let  $G$  be a finite group and  $V$  a faithful representation of  $G$  over  $\mathbb{C}$ . Then there is a natural action of  $G$  upon the field of rational functions  $\mathbb{C}(V)$ . A problem posed by Emmy Noether [25] asks as to whether the field of  $G$ -invariant functions  $\mathbb{C}(V)^G$  is purely transcendental over  $\mathbb{C}$ , i.e., whether the quotient space  $V/G$  is *rational*. A question related to the above mentioned is whether  $V/G$  is *stably rational*, that is, whether there exist independent variables  $x_1, \dots, x_r$  such that  $\mathbb{C}(V)^G(x_1, \dots, x_r)$  becomes a pure transcendental extension of  $\mathbb{C}$ . This problem has close connection with Lüroth's problem [27] and the inverse Galois problem [32, 28]. By Hilbert's Theorem 90 stable rationality of  $V/G$  does not depend upon the choice of  $V$ , but only on the group  $G$ . Saltman [28] found examples of groups  $G$  such that  $V/G$  is not stably rational over  $\mathbb{C}$ . His main method was application of the unramified cohomology group  $H_{\text{nr}}^2(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$  as an obstruction. A version of this invariant had been used before by Artin and Mumford [1] who constructed unirational varieties over  $\mathbb{C}$  that were not rational. Bogomolov [3] further explored this cohomology group. He proved that  $H_{\text{nr}}^2(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$  is canonically isomorphic to a certain subgroup  $B_0(G)$  (defined in Section 3) of the *Schur multiplier*  $H^2(G, \mathbb{Q}/\mathbb{Z})$  of  $G$ . Kunyavskiĭ [18] coined the term the *Bogomolov multiplier* of  $G$  for the group  $B_0(G)$ . Bogomolov used the above description to find new examples of groups with  $H_{\text{nr}}^2(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z}) \neq 0$ . Subsequently, Bogomolov, Maciel and Petrov [4] showed that  $B_0(G) = 0$  when  $G$  is a finite simple group of Lie type  $A_\ell$ , whereas Kunyavskiĭ [18] recently proved that  $B_0(G) = 0$  for every quasisimple or almost simple group  $G$ . Bogomolov's conjecture that  $V/G$  is stably rational over  $\mathbb{C}$  for every finite simple group  $G$ , nevertheless, still remains open.

We first observe that if  $G$  is a finite group, then  $B_0(G)$  is canonically isomorphic to  $\text{Hom}(\tilde{B}_0(G), \mathbb{Q}/\mathbb{Z})$ , where the group  $\tilde{B}_0(G)$  can be described as a section of the

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*nonabelian exterior square*  $G \wedge G$  of the group  $G$ . The latter appears implicitly in Miller's work [21], and was further developed by Brown and Loday [6]. Let  $\gamma_2(G)$  be the derived subgroup of  $G$ , and denote the kernel of the commutator homomorphism  $G \wedge G \rightarrow \gamma_2(G)$  by  $M(G)$ . Miller [21] proved that there is a natural isomorphism between  $M(G)$  and  $H_2(G, \mathbb{Z})$ . Using this description, we prove that  $\tilde{B}_0(G) = M(G)/M_0(G)$ , where  $M_0(G)$  is the subgroup of  $M(G)$  generated by all  $x \wedge y$  such that  $x, y \in G$  commute. In the finite case,  $\tilde{B}_0(G)$  is thus (non-canonically) isomorphic to  $B_0(G)$ . The functor  $\tilde{B}_0$  can be studied within the category of all groups, and this is the main goal of the paper. In the first part we prove a Hopf-type formula for  $\tilde{B}_0(G)$  by showing that if  $G$  is given by a free presentation  $G = F/R$ , then

$$\tilde{B}_0(G) \cong \frac{\gamma_2(F) \cap R}{\langle K(F) \cap R \rangle},$$

where  $K(F)$  denotes the set of commutators in  $F$ . A special case of this was implicitly used before by Bogomolov [3], and Bogomolov, Maciel and Petrov [4]. With the help of the above formula we derive a five term exact sequence

$$\tilde{B}_0(G) \longrightarrow \tilde{B}_0(G/N) \longrightarrow \frac{N}{\langle K(G) \cap N \rangle} \longrightarrow G^{\text{ab}} \longrightarrow (G/N)^{\text{ab}} \longrightarrow 0,$$

where  $G$  is any group and  $N$  a normal subgroup of  $G$ . This is a direct analogue of the well-known five term homological sequence. By applying Kunyavskii's work and the above sequence we obtain the following group theoretical result: If  $G$  is a finite group and  $S$  its solvable radical, then  $S \cap \gamma_2(G) = \langle S \cap K(G) \rangle$ . Furthermore, we compute  $\tilde{B}_0(G)$  when  $G$  is a finite group that is a split extension. This corresponds to a well-known result of Tahara [33] who computed the Schur multiplier of semidirect product of groups (see also [17]). In particular, we obtain a closed formula for  $B_0(G)$  when  $G$  is a Frobenius group.

In his paper [3], Bogomolov extended the definition of  $B_0(G)$  to cover all algebraic groups  $G$ , cf. Section 3. This can be further extended in a natural way to cover all infinite groups. We prove here that if  $G$  is any group, then  $B_0(G)$  is canonically isomorphic to  $\text{Hom}(\tilde{B}_{0\mathcal{F}}(G), \mathbb{Q}/\mathbb{Z})$ , where  $\tilde{B}_{0\mathcal{F}}(G)$  is the quotient of the subgroup  $M_{\mathcal{F}}(G)$  of  $H_2(G, \mathbb{Z})$  generated by all images of corestriction maps  $\text{cor}_G^H : H_2(H, \mathbb{Z}) \rightarrow H_2(G, \mathbb{Z})$ , where  $H$  runs through all finite subgroups of  $G$ , by the subgroup  $M_{0\mathcal{F}}(G)$  generated by all  $\text{im cor}_G^A$ , where  $A$  runs through all finite abelian subgroups of  $G$ . As a consequence we show that if  $G$  is a locally finite group, then  $\tilde{B}_0(G) \cong \tilde{B}_{0\mathcal{F}}(G)$ . This in particular applies to periodic linear groups. On the other hand,  $\tilde{B}_0(G)$  and  $\tilde{B}_{0\mathcal{F}}(G)$  may fail to be isomorphic in general.

One of the goals of the paper is to exhibit the role of  $\tilde{B}_0(G)$  in studying certain types of central extensions of  $G$ . This is motivated by the classical theory of Schur multipliers which are the cornerstones of the extension theory of groups. We define  $G \wr G$  to be the quotient of  $G \wedge G$  by  $M_0(G)$ . Then it is clear that the sequence  $\tilde{B}_0(G) \twoheadrightarrow G \wr G \twoheadrightarrow \gamma_2(G)$  is exact, therefore  $\tilde{B}_0(G)$  can be thought of as the obstruction to  $G \wr G$  being isomorphic to  $\gamma_2(G)$ . This corresponds to a result of Miller [21] who demonstrated that the nonabelian exterior square  $G \wedge G$  of a group  $G$  fits into the short exact sequence  $M(G) \twoheadrightarrow G \wedge G \twoheadrightarrow \gamma_2(G)$ . This construction enables us to prove that if  $G$  is a finite group, then for every stem extension  $(E, \pi, A)$  producing  $\tilde{B}_0(G)$  we have that  $\gamma_2(E)$  and  $G \wr G$  are of the same order. Furthermore, there exists a stem extension of this kind such that  $\gamma_2(E)$  is actually isomorphic to  $G \wr G$ . This can be seen as a direct analogue of the well-known fact that if  $G$  is finite, then  $G \wedge G$  is naturally isomorphic to the derived subgroup of an arbitrary covering group of  $G$ . In addition to that, we prove that if  $G$  is a perfect group, then  $G \wr G$  is universal within the class of central extensions  $E$  of  $G$  with the property

that every commuting pair of elements in  $G$  has commuting lifts in  $E$ . Again, this corresponds to the fact that if  $G$  is a perfect group, then  $G \wedge G$  is the universal central extension of  $G$ .

The first known examples of finite groups  $G$  with  $B_0(G) \neq 0$  were found among  $p$ -groups of class 2 [3, 28]. Bogomolov obtained a description of  $B_0(G)$  when  $G$  is a  $p$ -group of class 2 with  $G^{\text{ab}}$  elementary abelian. Here we obtain a description of  $\tilde{B}_0(G)$  for any group  $G$  that is nilpotent of class 2. More precisely, we show that  $\tilde{B}_0(G) \cong \ker(H_2(G^{\text{ab}}, \mathbb{Z}) \rightarrow \gamma_2(G)) / \ker(H_2(G^{\text{ab}}, \mathbb{Z}) \rightarrow G \wedge G)$ . In the case when  $G$  is a  $p$ -group of class 2 with  $G^{\text{ab}}$  elementary abelian, this can be further refined using the Blackburn-Evens theory [2].

The functor  $\tilde{B}_0$  has applications in K-theory. For a unital ring  $\Lambda$  define  $\tilde{B}_0 \Lambda = \tilde{B}_0(E(\Lambda))$  where  $E(\Lambda)$  is the subgroup of  $GL(\Lambda)$  generated by elementary matrices. We prove that  $\tilde{B}_0 \Lambda$  is naturally isomorphic to  $K_2 \Lambda / \langle K(\text{St}(\Lambda) \cap K_2 \Lambda) \rangle$ , where  $\text{St}(\Lambda)$  is the Steinberg group. This is related to a conjecture posed by Bass [10, Problem 3] that  $K_2 \Lambda$  is always generated by the so-called Milnor elements. We show that this problem has a positive solution for a ring  $\Lambda$  if and only if  $\tilde{B}_0 \Lambda$  is trivial. The latter is for instance true for commutative semilocal rings. A possible approach towards solving Bass' problem could be based on the result that  $\tilde{B}_0 \Lambda$  is naturally isomorphic to  $\tilde{B}_0(GL(\Lambda))$ .

In general it is hard to compute  $B_0(G)$ , due to its cohomological description. Chu, Hu, Kang, and Kunyavskii [7] recently completed calculations of  $B_0(G)$  for all groups of order  $\leq 64$ . The homological nature of  $\tilde{B}_0$ , on the other hand, allows machine computation of  $\tilde{B}_0(G)$  for polycyclic groups  $G$ . There is an efficient algorithm developed recently by Eick and Nickel [11] for computing  $G \wedge G$  in case  $G$  is polycyclic. Based on that we develop and implement an algorithm for computing  $\tilde{B}_0(G)$  for finite solvable groups  $G$ . We use this algorithm to determine the Bogomolov multiplier of all solvable groups of order  $\leq 729$ , apart from the orders 512, 576 and 640. Our computations in particular show that there exist three groups of order 243 with nontrivial unramified Brauer group. This contradicts a result of Bogomolov [3] claiming that if  $G$  is a finite  $p$ -group of order at most  $p^5$ , then  $B_0(G) = 0$ .

## 2. PRELIMINARIES AND NOTATIONS

In this section we fix some notations used throughout the paper. Let  $G$  be a group and  $x, y \in G$ . We use the notation  ${}^x y = xyx^{-1}$  for conjugation from the left. The commutator  $[x, y]$  of elements  $x$  and  $y$  is defined by  $[x, y] = xyx^{-1}y^{-1} = {}^x y y^{-1}$ . If  $H$  and  $K$  are subgroups of  $G$ , then we define  $[H, K] = \langle [h, k] \mid h \in H, k \in K \rangle$ . The commutator subgroup  $\gamma_2(G)$  of  $G$  is defined to be the group  $[G, G]$ . The set  $\{[x, y] \mid x, y \in G\}$  of all commutators of  $G$  is denoted by  $K(G)$ .

We recall the definition and basic properties of the nonabelian exterior product of groups. The reader is referred to [6, 21] for more thorough accounts on the theory and its generalizations. Let  $G$  be a group and  $M$  and  $N$  normal subgroups of  $G$ . We form the group  $M \wedge N$ , generated by the symbols  $m \wedge n$ , where  $m \in M$  and  $n \in N$ , subject to the following relations:

$$(2.0.1) \quad mm' \wedge n = ({}^m m' \wedge {}^m n)(m \wedge n),$$

$$(2.0.2) \quad m \wedge nn' = (m \wedge n)({}^n m \wedge {}^n n'),$$

$$(2.0.3) \quad x \wedge x = 1,$$

for all  $m, m' \in M, n, n' \in N$  and  $x \in M \cap N$ .

Let  $L$  be a group. A function  $\phi : M \times N \rightarrow L$  is called a *crossed pairing* if for all  $m, m' \in M, n, n' \in N$ ,  $\phi(mm', n) = \phi({}^m m', {}^m n)\phi(m, n)$ ,  $\phi(m, nn') =$

$\phi(m, n)\phi({}^n m, {}^n n')$ , and  $\phi(x, x) = 1$  for all  $x \in M \cap N$ . A crossed pairing  $\phi$  determines a unique homomorphism of groups  $\phi^* : M \wedge N \rightarrow L$  such that  $\phi^*(m \wedge n) = \phi(m, n)$  for all  $m \in M, n \in N$ .

The group  $G \wedge G$  is said to be the *nonabelian exterior square* of  $G$ . By definition, the commutator map  $\kappa : G \wedge G \rightarrow \gamma_2(G)$ , given by  $g \wedge h \mapsto [g, h]$ , is a well defined homomorphism of groups. Clearly  $M(G) = \ker \kappa$  is central in  $G \wedge G$ , and  $G$  acts trivially via diagonal action on  $M(G)$ . Miller [21] proved that there is a natural isomorphism between  $M(G)$  and  $H_2(G, \mathbb{Z})$ . A direct consequence of this result is that if a group  $G$  is given by a free presentation  $G \cong F/R$ , then  $G \wedge G$  is naturally isomorphic to  $\gamma_2(F)/[R, F]$ .

The following lemma collects some basic identities that hold in the nonabelian exterior square of a group:

**Lemma 2.1** ([6]). *Let  $G$  be a group and  $x, y, z, w \in G$ .*

- (a)  $x \wedge y = (y \wedge x)^{-1}$ .
- (b)  $x^{-1}(x \wedge y) = y \wedge x^{-1}$ .
- (c)  $[z, w](x \wedge y) = (z \wedge w)(x \wedge y)(z \wedge w)^{-1}$ .

### 3. THE UNRAMIFIED BRAUER GROUP

Let  $G$  be a finite group and  $V$  a faithful representation of  $G$  over  $\mathbb{C}$ . Bogomolov [3] proved that the unramified Brauer group  $H_{\text{nr}}^2(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$  is canonically isomorphic to the group

$$(3.0.1) \quad B_0(G) = \bigcap_{\substack{A \leq G, \\ A \text{ abelian}}} \ker \text{res}_A^G,$$

where  $\text{res}_A^G : H^2(G, \mathbb{Q}/\mathbb{Z}) \rightarrow H^2(A, \mathbb{Q}/\mathbb{Z})$  is the usual cohomological restriction map. Our first aim is to obtain a homological description of  $B_0(G)$ . Thus we need a dual of the above construction. Let  $H$  be a subgroup of  $G$ . Then there is a corestriction map  $\text{cor}_G^H : H_2(H, \mathbb{Z}) \rightarrow H_2(G, \mathbb{Z})$ . On the other hand, we have a natural map  $\tau_G^H : H \wedge H \rightarrow G \wedge G$ . Identifying  $H_2(G, \mathbb{Z})$  with  $M(G)$  and  $H_2(H, \mathbb{Z})$  with  $M(H)$ , we can write  $\text{cor}_G^H = \tau_G^H|_{M(H)}$ . Thus we have the following commutative diagram with exact rows:

$$(3.0.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H_2(H, \mathbb{Z}) & \longrightarrow & H \wedge H & \longrightarrow & \gamma_2(H) \longrightarrow 1 \\ & & \downarrow \text{cor}_G^H & & \downarrow \tau_G^H & & \downarrow \\ 0 & \longrightarrow & H_2(G, \mathbb{Z}) & \longrightarrow & G \wedge G & \longrightarrow & \gamma_2(G) \longrightarrow 1 \end{array}$$

Now define

$$M_0(G) = \langle \text{cor}_G^A M(A) \mid A \leq G, A \text{ abelian} \rangle.$$

This group can be described as a subgroup of  $G \wedge G$  in the following way.

**Lemma 3.1.** *Let  $G$  be a group. Then*

$$M_0(G) = \langle x \wedge y \mid x, y \in G, [x, y] = 1 \rangle.$$

*Proof.* Denote  $N = \langle x \wedge y \mid x, y \in G, [x, y] = 1 \rangle$ . Suppose that  $x, y \in G$  commute. Then  $A = \langle x, y \rangle$  is an abelian subgroup of  $G$ , hence  $\text{cor}_G^A M(A) \leq M_0(G)$ . In particular,  $x \wedge y \in M_0(G)$ .

Conversely, let  $A$  be an abelian subgroup of  $G$ . Let  $w \in \text{cor}_G^A M(A)$ . Then  $w$  can be written as

$$w = \prod_{i=1}^r (a_i \wedge b_i),$$

where  $a_i, b_i \in A$ . Since  $[a_i, b_i] = 1$  for all  $i = 1, \dots, r$ , it follows that  $w \in N$ . This concludes the proof.  $\square$

For a group  $G$  denote

$$\tilde{B}_0(G) = M(G)/M_0(G).$$

With this notation we have the following result.

**Theorem 3.2.** *Let  $G$  be a finite group. Then  $B_0(G)$  is naturally isomorphic to  $\text{Hom}(\tilde{B}_0(G), \mathbb{Q}/\mathbb{Z})$ , and thus  $B_0(G) \cong \tilde{B}_0(G)$  (non-canonically).*

*Proof.* At first we describe the natural isomorphism between the Schur multiplier  $H^2(G, \mathbb{Q}/\mathbb{Z})$  and  $\text{Hom}(H_2(G, \mathbb{Z}), \mathbb{Q}/\mathbb{Z})$  in terms of the nonabelian exterior square of  $G$ . Choose  $\gamma \in H^2(G, \mathbb{Q}/\mathbb{Z})$  and let

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{i} G_\gamma \xrightarrow{\pi} G \longrightarrow 1$$

be the central extension associated to  $\gamma$ . Define a map  $G \times G \rightarrow \gamma_2(G_\gamma)$  by the rule  $(x, y) \mapsto [\bar{x}, \bar{y}]$ , where  $\bar{x}$  and  $\bar{y}$  are preimages in  $G_\gamma$  under  $\pi$  of  $x$  and  $y$ , respectively. This map is well defined. Furthermore, it is a crossed pairing, hence it induces a homomorphism  $\lambda_\gamma : G \wedge G \rightarrow \gamma_2(G_\gamma)$  given by  $\lambda_\gamma(x \wedge y) = [\bar{x}, \bar{y}]$  for  $x, y \in G$ . It is clear that if  $c \in M(G)$ , then  $\lambda_\gamma(c) \in i(\mathbb{Q}/\mathbb{Z})$ , therefore the restriction of  $\lambda_\gamma$  to  $M(G)$  (still denoted by  $\lambda_\gamma$ ) belongs to  $\text{Hom}(M(G), \mathbb{Q}/\mathbb{Z})$ . The map  $\Theta : H^2(G, \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Hom}(M(G), \mathbb{Q}/\mathbb{Z})$  given by  $\gamma \mapsto \lambda_\gamma$  is a homomorphism of groups.

Conversely, let  $\varphi \in \text{Hom}(M(G), \mathbb{Q}/\mathbb{Z})$ . Let  $H$  be a covering group of  $G$ . In other words, we have a central extension

$$0 \longrightarrow Z \xrightarrow{j} H \xrightarrow{\rho} G \longrightarrow 1$$

with  $jZ \leq \gamma_2(H)$  and  $Z \cong M(G)$ . Every finite group has at least one covering group by a result of Schur, cf. [15, Hauptsatz V.23.5]. By [6] we have that  $\gamma_2(H)$  is canonically isomorphic to  $G \wedge G$ . Upon identifying  $\gamma_2(H)$  with  $G \wedge G$ , we may assume without loss of generality that  $M(G)$  is a subgroup of  $H$ . Choose a section  $\mu : G \rightarrow H$  of  $\rho$  and define a map  $f : G \times G \rightarrow H$  by  $f(x, y) = \mu(x)\mu(y)\mu(xy)^{-1}$  for  $x, y \in G$ . It is straightforward to verify  $f$  maps  $G \times G$  into  $M(G)$ , and that  $\varphi f \in Z^2(G, \mathbb{Q}/\mathbb{Z})$ . The cohomology class of  $\varphi f$  does not depend upon the choice of  $\mu$ . We therefore have a map (the so-called transgression map)

$$\text{tra} : \text{Hom}(M(G), \mathbb{Q}/\mathbb{Z}) \rightarrow H^2(G, \mathbb{Q}/\mathbb{Z})$$

given by  $\text{tra}(\varphi) = [\varphi f]$ . This is easily seen to be a homomorphism, and  $\Theta$  is its inverse.

Now choose  $\gamma \in B_0(G)$  and let the map  $\Theta : H^2(G, \mathbb{Q}/\mathbb{Z}) \rightarrow \text{Hom}(M(G), \mathbb{Q}/\mathbb{Z})$  be defined as above. Denote  $\lambda_\gamma = \Theta(\gamma)$ . Let  $x, y \in G$  and suppose that  $[x, y] = 1$ . Then  $A = \langle x, y \rangle$  is an abelian subgroup of  $G$ , therefore  $\text{res}_A^G(\gamma) = 0$ . This implies that  $\lambda_\gamma(x \wedge y) = [\bar{x}, \bar{y}] = 1$ . Therefore  $\Theta$  induces a homomorphism  $\tilde{\Theta} : B_0(G) \rightarrow \text{Hom}(M(G)/M_0(G), \mathbb{Q}/\mathbb{Z})$ .

Let  $\varphi \in \text{Hom}(M(G)/M_0(G), \mathbb{Q}/\mathbb{Z})$ . Then  $\varphi$  can be lifted to a homomorphism  $\tilde{\varphi} : M(G) \rightarrow \mathbb{Q}/\mathbb{Z}$ . Put  $\gamma = \text{tra}(\tilde{\varphi})$ . Suppose that

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{i} G_\gamma \xrightarrow{\pi} G \longrightarrow 1$$

is a central extension associated to  $\gamma$ . Choose an arbitrary bicyclic subgroup  $A = \langle a, b \rangle$  of  $G$ . Then we have a central extension

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \xrightarrow{i} A_\gamma \xrightarrow{\pi|_{A_\gamma}} A \longrightarrow 1$$

that corresponds to  $\text{res}_A^G(\gamma)$ . Since  $[a, b] = 1$ , we have that  $a \wedge b \in M_0(G) \leq \ker \bar{\varphi}$ , therefore  $[\bar{a}, \bar{b}] = 1$  in  $A_\gamma$ . It follows that  $A_\gamma$  is abelian, thus  $\gamma \in B_0(G)$ . Hence the transgression map induces a homomorphism  $\widetilde{\text{tra}} : \text{Hom}(M(G)/M_0(G), \mathbb{Q}/\mathbb{Z}) \rightarrow B_0(G)$  whose inverse is  $\bar{\Theta}$ .  $\square$

The definition of  $B_0(G)$  can be extended to infinite groups as follows [3]. Let  $G$  be a group. Define

$$K_G = \{\gamma \in H^2(G, \mathbb{Q}/\mathbb{Z}) \mid \text{res}_H^G \gamma = 0 \text{ for every finite } H \leq G\}.$$

Let  $B_0(G)$  be the subgroup of  $H^2(G, \mathbb{Q}/\mathbb{Z})/K_G$  consisting of all  $\gamma + K_G$  with the property that  $\text{res}_A^G \gamma = 0$  for every finite abelian subgroup  $A$  of  $G$ . It is clear that if  $G$  is a finite group, then this definition of  $B_0(G)$  coincides with the one given by (3.0.1). Bogomolov [3, Theorem 3.1] showed that if  $G$  is an algebraic group, then  $B_0(G)$  is isomorphic to  $H_{\text{nr}}^2(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$ , where  $V$  is any generically free representation of  $G$ .

In order to obtain a homological description of  $B_0(G)$  for infinite groups, we denote

$$M_{\mathcal{F}}(G) = \langle \text{cor}_G^H M(H) \mid H \leq G, |H| < \infty \rangle$$

and

$$M_{0\mathcal{F}}(G) = \langle \text{cor}_G^A M(A) \mid A \leq G, |A| < \infty, A \text{ abelian} \rangle.$$

Note that a similar argument as that of Lemma 3.1 shows that  $M_{0\mathcal{F}}(G) = \langle x \wedge y \mid [x, y] = 1, |x| < \infty, |y| < \infty \rangle$ . Now define  $\tilde{B}_{0\mathcal{F}}(G) = M_{\mathcal{F}}(G)/M_{0\mathcal{F}}(G)$ . Then we have:

**Theorem 3.3.** *Let  $G$  be a group. Then the group  $B_0(G)$  is naturally isomorphic to  $\text{Hom}(\tilde{B}_{0\mathcal{F}}(G), \mathbb{Q}/\mathbb{Z})$ .*

*Proof.* We have a natural isomorphism  $H^2(G, \mathbb{Q}/\mathbb{Z}) \cong \text{Hom}(M(G), \mathbb{Q}/\mathbb{Z})$ . For  $\gamma \in H^2(G, \mathbb{Q}/\mathbb{Z})$  denote by  $\lambda_\gamma$  the corresponding element of  $\text{Hom}(M(G), \mathbb{Q}/\mathbb{Z})$ . By our definition we have that  $\gamma \in K_G$  if and only if  $M_{\mathcal{F}}(G) \leq \ker \lambda_\gamma$ . Therefore  $H^2(G, \mathbb{Q}/\mathbb{Z})/K_G$  is naturally isomorphic to  $\text{Hom}(M_{\mathcal{F}}(G), \mathbb{Q}/\mathbb{Z})$ . Adapting the argument of the proof of Theorem 3.2, we obtain the required result.  $\square$

In general,  $\tilde{B}_0(G)$  and  $\tilde{B}_{0\mathcal{F}}(G)$  may be quite different. For example, if  $G$  is a one-relator group with torsion, then  $G$  has a presentation  $G = \langle X \mid s^m \rangle$ , where  $s$  is not a proper power in the free group over  $X$ . By a result of Newman [24], every finite subgroup of  $G$  is conjugate to a subgroup of  $\langle s \rangle$ , hence  $\tilde{B}_{0\mathcal{F}}(G) = 0$ . On the other hand, all centralizers of nontrivial elements of  $G$  are cyclic [24], hence  $M_0(G) = 0$  and therefore  $\tilde{B}_0(G) \cong H_2(G, \mathbb{Z})$ . The latter can be nontrivial, cf. Lyndon [19].

In the case of locally finite groups we have the following:

**Corollary 3.4.** *Let  $G$  be a locally finite group. Then  $\tilde{B}_0(G) \cong \tilde{B}_{0\mathcal{F}}(G)$ .*

*Proof.* Every group  $G$  is a direct limit of its finitely generated subgroups  $\{G_\lambda \mid \lambda \in \Lambda\}$ . If  $G$  is locally finite, then the groups  $G_\lambda$  are all finite. Since  $M(G) \cong \varinjlim M(G_\lambda)$ , we conclude that  $M(G) = M_{\mathcal{F}}(G)$ . Since  $G$  is periodic, we also have that  $M_{0\mathcal{F}}(G) = M_0(G)$ , hence the result.  $\square$

Corollary 3.4 applies, for example, to periodic linear groups. On the other hand, there exist finitely generated periodic groups  $G$  (even of finite exponent) such that  $\tilde{B}_{0\mathcal{F}}(G) = 0$ , yet  $\tilde{B}_0(G)$  is nontrivial.

*Example 3.5.* Suppose  $m > 1$  and let  $n > 2^{48}$  be odd. Let  $F$  be a free group of rank  $m$ . Denote  $B(m, n) = F/F^n$ , the *free Burnside group* of rank  $m$  and exponent  $n$ . Ivanov [16] showed that all centralizers of nontrivial elements of  $B(m, n)$  are cyclic, and that every finite subgroup of  $B(m, n)$  is cyclic. From here it follows

that  $\tilde{B}_{0\mathcal{F}}(B(m, n)) = 0$  and  $\tilde{B}_0(B(m, n)) \cong H_2(B(m, n), \mathbb{Z})$ . The latter group is free abelian of countable rank, cf. [26, Corollary 31.2].

In the rest of the paper we mainly consider the properties of  $\tilde{B}_0(G)$ . Obviously  $\tilde{B}_0$  is a covariant functor from **Gr** to **Ab**. It is well known that the homology functor commutes with direct limits. It turns out that  $\tilde{B}_0$  enjoys the same property:

**Proposition 3.6.** *The functor  $\tilde{B}_0$  commutes with direct limits. More precisely, if  $\{G_\lambda, \alpha_\lambda^\mu \mid \lambda \leq \mu \in \Lambda\}$  is a direct system of groups and  $G$  its direct limit, then  $\tilde{B}_0(G)$  is the direct limit of  $\{\tilde{B}_0(G_\lambda), \tilde{B}_0(\alpha_\lambda^\mu) \mid \lambda \leq \mu \in \Lambda\}$ .*

*Proof.* For every  $\lambda \in \Lambda$  we have

$$0 \longrightarrow M_0(G_\lambda) \longrightarrow M(G_\lambda) \longrightarrow \tilde{B}_0(G_\lambda) \longrightarrow 0,$$

hence the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \varinjlim M_0(G_\lambda) & \longrightarrow & \varinjlim M(G_\lambda) & \longrightarrow & \varinjlim \tilde{B}_0(G_\lambda) \longrightarrow 0 \\ & & \alpha' \downarrow & & \alpha \downarrow & & \tilde{\alpha} \downarrow \\ 0 & \longrightarrow & M_0(G) & \longrightarrow & M(G) & \longrightarrow & \tilde{B}_0(G) \longrightarrow 0 \end{array}$$

is commutative with exact rows. Here  $\alpha$  is the natural isomorphism, and  $\alpha'$  is its restriction. Clearly  $\alpha'$  is an isomorphism, hence so is  $\tilde{\alpha}$ .  $\square$

**Proposition 3.7.** *Let  $G_1$  and  $G_2$  be groups. Then  $\tilde{B}_0(G_1 * G_2) \cong \tilde{B}_0(G_1) \times \tilde{B}_0(G_2)$  and  $\tilde{B}_{0\mathcal{F}}(G_1 * G_2) \cong \tilde{B}_{0\mathcal{F}}(G_1) \times \tilde{B}_{0\mathcal{F}}(G_2)$ .*

*Proof.* Let  $G = G_1 * G_2$  and let  $\iota_1 : G_1 \rightarrow G$  and  $\iota_2 : G_2 \rightarrow G$  be the canonical injections. Then the induced maps  $\iota_i^* : M(G_i) \rightarrow M(G)$  are injective ( $i = 1, 2$ ),  $\iota_1^* M(G_1) \cap \iota_2^* M(G_2) = 1$  and  $M(G) = \iota_1^* M(G_1) \times \iota_2^* M(G_2)$  by [21]. Now let  $a, b \in G \setminus \{1\}$  with  $[a, b] = 1$ . By [20, p. 196] we have the following possibilities. If  $a \in {}^h\iota_1(G_1)$ , then  $b \in C_G(a) \leq {}^h\iota_1(G_1)$ , hence we can write  $a = {}^h\iota_1(x)$  and  $b = {}^h\iota_1(y)$  for some commuting elements  $x, y \in G_1$ . In this case we get  $a \wedge b = {}^h(\iota_1(x) \wedge \iota_1(y)) = \iota_1(x) \wedge \iota_1(y)$ , as  $G$  acts trivially on  $M(G)$ . For  $a \in {}^h\iota_2(G_2)$ , the situation is similar. If neither  $a \in {}^h\iota_1(G_1)$  nor  $a \in {}^h\iota_2(G_2)$ ,  $C_G(a)$  is infinite cyclic. In this case we clearly have that  $a \wedge b = 1$ . Therefore we conclude that  $M_0(G) = \iota_1^* M_0(G_1) \times \iota_2^* M_0(G_2)$ . It follows from here that

$$\tilde{B}_0(G) \cong \iota_1^\# \tilde{B}_0(G_1) \times \iota_2^\# \tilde{B}_0(G_2),$$

where  $\iota_i^\# : \tilde{B}_0(G_i) \rightarrow \tilde{B}_0(G)$  are the maps induced by  $\iota_i$ ,  $i = 1, 2$ . From the diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & M_0(G_i) & \longrightarrow & M(G_i) & \longrightarrow & \tilde{B}_0(G_i) \longrightarrow 1 \\ & & \iota_i^* \downarrow & & \iota_i^* \downarrow & & \iota_i^\# \downarrow \\ 1 & \longrightarrow & M_0(G) & \longrightarrow & M(G) & \longrightarrow & \tilde{B}_0(G) \longrightarrow 1 \end{array}$$

we see that  $\iota_i^\#$ ,  $i = 1, 2$ , are both injective, therefore  $\tilde{B}_0(G_1 * G_2) \cong \tilde{B}_0(G_1) \times \tilde{B}_0(G_2)$ .

It remains to prove the corresponding assertion for  $\tilde{B}_{0\mathcal{F}}(G_1 * G_2)$ . The above argument shows that  $M_{0\mathcal{F}}(G) = \iota_1^* M_{0\mathcal{F}}(G_1) \times \iota_2^* M_{0\mathcal{F}}(G_2)$ . By [5, p. 54], every finite subgroup of  $G$  is conjugate to a subgroup of  $G_1$  or  $G_2$ . Since  $G$  acts trivially on  $M(G)$ , we therefore conclude that  $M_{\mathcal{F}}(G) = \langle \text{cor}_G^H M(H) \mid H \leq G_1 \text{ or } H \leq G_2, |H| < \infty \rangle = \iota_1^* M_{\mathcal{F}}(G_1) \times \iota_2^* M_{\mathcal{F}}(G_2)$ . From here the result follows along the same lines as above.  $\square$

Let the group  $G$  be given by a free presentation  $G = F/R$ , where  $F$  is a free group and  $R$  a normal subgroup of  $F$ . By the well known Hopf formula [5, Theorem II.5.3] we have that  $M(G) \cong (\gamma_2(F) \cap R)/[R, F]$ . The isomorphism is induced by the canonical isomorphism  $G \wedge G \rightarrow \gamma_2(F)/[R, F]$  given by  $xR \wedge yR \mapsto [x, y][R, F]$ . Under this map,  $M_0(G)$  can be identified with the subgroup of  $F/[F, R]$  generated by all the commutators in  $F/[F, R]$  that belong to the Schur multiplier of  $G$ . In other words, we have that  $M_0(G) \cong \langle K(F/[R, F]) \cap R/[R, F] \rangle = \langle K(F) \cap R \rangle [R, F]/[R, F] = \langle K(F) \cap R \rangle / [R, F]$ . Thus we have proved the following Hopf-type formula for  $\tilde{B}_0(G)$ :

**Proposition 3.8.** *Let  $G$  be a group given by a free presentation  $G = F/R$ . Then*

$$\tilde{B}_0(G) \cong \frac{\gamma_2(F) \cap R}{\langle K(F) \cap R \rangle}.$$

This formula enables, in principle, explicit calculations of  $\tilde{B}_0(G)$ , given a free presentation of  $G$ . For example, a word  $w$  in a free group  $F$  is said to be a *commutator word* if  $w = [u, v]$  for some  $u, v \in F$ . We have the following result:

**Corollary 3.9.** *Let  $\mathfrak{V}$  be a variety of groups defined by a commutator word  $w$ . If  $G$  is a  $\mathfrak{V}$ -relatively free group, then  $\tilde{B}_0(G) = 0$ .*

*Proof.* Let  $w$  be an  $n$ -variable commutator word.  $G$  can be presented as a quotient  $F/\mathfrak{V}(F)$  of a free group  $F$  by the verbal subgroup  $\mathfrak{V}(F) = \langle w(f_1, \dots, f_n) \mid f_1, \dots, f_n \in F \rangle$  of  $F$ . Note that  $\mathfrak{V}(F) \leq \gamma_2(F)$  and  $\langle K(F) \cap \mathfrak{V}(F) \rangle = \mathfrak{V}(F)$ . By Proposition 3.8 we get the result.  $\square$

On the other hand, there exist relatively free groups  $G$  with  $\tilde{B}_0(G) \neq 0$ , cf. Example 3.5 and Section 8.

Another interpretation of  $\tilde{B}_0(G)$  for finite groups  $G$  can be obtained via covering groups. Covering groups of a given group  $G = F/R$  may not be unique, yet their derived subgroups are all naturally isomorphic to  $\gamma_2(F)/[R, F]$ . Under this identification we have the following result.

**Proposition 3.10.** *Let  $G$  be a finite group and  $H$  its covering group. Let  $Z$  be a central subgroup of  $H$  such that  $Z \leq \gamma_2(H)$ ,  $Z \cong M(G)$  and  $H/Z \cong G$ . Then*

$$\tilde{B}_0(G) \cong \frac{Z}{\langle K(H) \cap Z \rangle}.$$

*In particular,  $\tilde{B}_0(G) = 0$  if and only if every element of  $Z$  can be represented as a product of commutators that all belong to  $Z$ .*

We note here that a special case of Proposition 3.10 formed one of the crucial steps in proving the main results of [4] and [18].

One of the main features of the homological description of  $B_0(G)$  is a five term exact sequence associated to the short exact sequence  $1 \rightarrow N \rightarrow G \rightarrow G/N \rightarrow 1$  of groups. This sequence is an unramified Brauer group analogue of the well known five term homology sequence, cf [5, p. 46].

**Theorem 3.11.** *Let  $G$  be a group and  $N$  a normal subgroup of  $G$ . Then we have the following exact sequence:*

$$\tilde{B}_0(G) \longrightarrow \tilde{B}_0(G/N) \longrightarrow \frac{N}{\langle K(G) \cap N \rangle} \longrightarrow G^{\text{ab}} \longrightarrow (G/N)^{\text{ab}} \longrightarrow 0.$$

*Proof.* Let  $G$  have a free presentation  $G = F/R$ , and let  $SR/R$  be the corresponding free presentation of  $N$ . Then Proposition 3.8 implies that  $\tilde{B}_0(G) \cong (\gamma_2(F) \cap R)/\langle K(F) \cap R \rangle$  and  $\tilde{B}_0(G/N) \cong (\gamma_2(F) \cap RS)/\langle K(F) \cap RS \rangle$ . The canonical



epimorphism  $\rho : G \rightarrow G/N$  induces a homomorphism  $\rho^\# : \tilde{B}_0(G) \rightarrow \tilde{B}_0(G/N)$ . From the above Hopf formulae it follows that

$$\ker \rho^\# = \frac{R \cap \langle K(F) \cap RS \rangle}{\langle K(F) \cap R \rangle}$$

and

$$\operatorname{im} \rho^\# = \frac{\gamma_2(F) \cap \langle K(F) \cap RS \rangle R}{\langle K(F) \cap RS \rangle}.$$

It is straightforward to verify that  $\langle K(G) \cap N \rangle = \langle K(F) \cap RS \rangle R/R$ , therefore  $N/\langle K(G) \cap N \rangle \cong RS/\langle K(F) \cap RS \rangle R$ . Thus there is a natural map  $\sigma : \tilde{B}_0(G/N) \rightarrow N/\langle K(G) \cap N \rangle$ . We have that  $\ker \sigma = \operatorname{im} \rho^\#$  and

$$\operatorname{im} \sigma = \frac{(\gamma_2(F) \cap RS)R}{\langle K(F) \cap RS \rangle R} = \frac{\gamma_2(F)R \cap RS}{\langle K(F) \cap RS \rangle R} = \frac{\gamma_2(G) \cap N}{\langle K(G) \cap N \rangle}.$$

Furthermore, there is a natural map  $\pi : N/\langle K(G) \cap N \rangle \rightarrow G^{\text{ab}}$  whose kernel is equal to  $\operatorname{im} \sigma$ , and  $\operatorname{im} \pi = N\gamma_2(G)/\gamma_2(G)$ . Finally, there is a surjective homomorphism  $G^{\text{ab}} \rightarrow (G/N)^{\text{ab}}$  whose kernel is equal to  $\operatorname{im} \pi$ . From here our assertion readily follows.  $\square$

The proof of Theorem 3.11 also yields another exact sequence that is an analogue of the corresponding sequence for Schur multipliers obtained by Blackburn and Evens [2]. More precisely, we have:

**Proposition 3.12.** *Let  $G$  be a group given by a free presentation  $G = F/R$  and let  $N = SR/R$  be a normal subgroup of  $G$ . Then the sequence*

$$0 \rightarrow \frac{R \cap \langle K(F) \cap RS \rangle}{\langle K(F) \cap R \rangle} \rightarrow \tilde{B}_0(G) \rightarrow \tilde{B}_0(G/N) \rightarrow \frac{N \cap \gamma_2(G)}{\langle K(G) \cap N \rangle} \rightarrow 0$$

*is exact.*

The above result has the following group theoretical consequence:

**Corollary 3.13.** *Let  $G$  be a finite group and  $S$  the solvable radical of  $G$ , i.e., the largest solvable normal subgroup of  $G$ . Then  $S \cap \gamma_2(G) = \langle S \cap K(G) \rangle$ .*

*Proof.* The factor group  $G/S$  does not contain proper nontrivial abelian normal subgroups, i.e., it is semisimple. By a result of Kunyavskii [18] we conclude that  $\tilde{B}_0(G/S) = 0$ . From Proposition 3.12 we get the desired result.  $\square$

#### 4. THE ‘COMMUTATIVITY-PRESERVING’ NONABELIAN EXTERIOR PRODUCT OF GROUPS

The nonabelian exterior square of a group encodes crucial information on the Schur multiplier of the group. In this section we introduce a related construction that plays a similar role when considering the functor  $\tilde{B}_0$ .

Let  $G$  be a group and  $M$  and  $N$  normal subgroups of  $G$ . We form the group  $M \rtimes N$ , generated by the symbols  $m \rtimes n$ , where  $m \in M$  and  $n \in N$ , subject to the following relations:

$$\begin{aligned} (4.0.1) \quad & mm' \rtimes n = ({}^m m' \rtimes {}^m n)(m \rtimes n), \\ & m \rtimes nn' = (m \rtimes n)({}^n m \rtimes {}^n n'), \\ & x \rtimes y = 1, \end{aligned}$$

for all  $m, m' \in M$ ,  $n, n' \in N$ , and all  $x \in M$  and  $y \in N$  with  $[x, y] = 1$ . If we denote  $M_0(M, N) = \langle m \rtimes n \mid m \in M, n \in N, [m, n] = 1 \rangle$ , then we have that  $M \rtimes N = (M \rtimes N)/M_0(M, N)$ .

Let  $L$  be a group. A function  $\phi : M \times N \rightarrow L$  is called a  $\tilde{B}_0$ -pairing if for all  $m, m' \in M$ ,  $n, n' \in N$ , and for all  $x \in M$ ,  $y \in N$  with  $[x, y] = 1$ ,

$$\begin{aligned}\phi(mm', n) &= \phi({}^m m', {}^m n)\phi(m, n), \\ \phi(m, nn') &= \phi(m, n)\phi({}^n m, {}^n n'), \\ \phi(x, y) &= 1.\end{aligned}$$

Clearly a  $\tilde{B}_0$ -pairing  $\phi$  determines a unique homomorphism of groups  $\phi^* : M \rtimes N \rightarrow L$  such that  $\phi^*(m \rtimes n) = \phi(m, n)$  for all  $m \in M$ ,  $n \in N$ . An example of a  $\tilde{B}_0$ -pairing is the commutator map  $M \times N \rightarrow [M, N]$ . It induces a homomorphism  $\tilde{\kappa} : M \rtimes N \rightarrow [M, N]$  such that  $\tilde{\kappa}(m \rtimes n) = [m, n]$  for all  $m \in M$  and  $n \in N$ . We denote the kernel of this homomorphism by  $\tilde{B}_0(M, N)$ .

In the case when  $M = N = G$ , we have that  $M_0(G, G) = M_0(G)$  and  $\tilde{B}_0(G, G) = \tilde{B}_0(G)$ . We therefore have a central extension

$$0 \longrightarrow \tilde{B}_0(G) \longrightarrow G \rtimes G \xrightarrow{\tilde{\kappa}} \gamma_2(G) \longrightarrow 1,$$

where  $\tilde{\kappa}$  is the commutator map. Thus one can interpret  $\tilde{B}_0(G)$  as a measure of the extent to which relations among commutators in  $G$  fail to be consequences of ‘universal’ commutator relations given by the images of relations (4.0.1) under the commutator map.

**Proposition 4.1.** *Let  $M$  and  $N$  be normal subgroups of a group  $G$ . Let  $K \leq M \cap N$  be a normal subgroup of  $G$ . Then  $M/K \rtimes N/K \cong (M \rtimes N)/J$ , where  $J = \langle m \rtimes n \mid m \in M, n \in N, [m, n] \in K \rangle$ .*

*Proof.* The map  $M/K \times N/K \rightarrow (M \rtimes N)/J$  given by  $(mK, nK) \mapsto (m \rtimes n)J$  is well defined and a  $\tilde{B}_0$ -pairing, hence it induces a homomorphism  $\varphi : M/K \rtimes N/K \rightarrow (M \rtimes N)/J$ . On the other hand, we have a canonical  $\tilde{B}_0$ -pairing  $M \times N \rightarrow M/K \rtimes N/K$  that induces a homomorphism  $M \rtimes N \rightarrow M/K \rtimes N/K$ . Under this homomorphism  $J$  gets mapped to 1, hence we have a homomorphism  $\psi : (M \rtimes N)/J \rightarrow M/K \rtimes N/K$  whose inverse is  $\varphi$ .  $\square$

Schur [29], cf. also [15, Kapitel V], developed the theory of stem extensions. Here we indicate the role  $\tilde{B}_0(G)$  and  $G \rtimes G$  within the theory. Let  $G$  be a finite group and denote by  $(E, \pi, A)$  the central extension

$$(4.1.1) \quad 1 \longrightarrow A \longrightarrow E \xrightarrow{\pi} G \longrightarrow 1$$

of  $G$ . If the transgression homomorphism  $\text{tra} : \text{Hom}(A, \mathbb{Q}/\mathbb{Z}) \rightarrow H^2(G, \mathbb{Q}/\mathbb{Z})$  is injective, then we say that  $(E, \pi, A)$  is a *stem extension* of  $G$ , and that the group  $B = \text{im tra}$  is *produced* by  $(E, \pi, A)$ . One can show that a central extension  $(E, \pi, A)$  of  $G$  is a stem extension if and only if  $A \leq \gamma_2(E)$ . In this case we have that  $B \cong A$ .

By a well known result of Schur [29], every subgroup of  $H^2(G, \mathbb{Q}/\mathbb{Z})$  is produced by some stem extension of  $G$ . This, in particular, applies to  $B_0(G)$ . In terms of its homological counterpart  $\tilde{B}_0(G)$ , we obtain the following result.

**Theorem 4.2.** *Let  $G$  be a finite group. Let  $(E, \pi, A)$  be a stem extension that produces  $\tilde{B}_0(G)$ . Then  $|\gamma_2(E)| = |G \rtimes G|$ . Furthermore, there exists a stem extension  $(E, \pi, A)$  of  $G$  producing  $\tilde{B}_0(G)$  such that  $\gamma_2(E) \cong G \rtimes G$ .*

*Proof.* Let  $G = F/R$  be a free presentation of  $G$ . By Proposition 4.1 we have that  $G \rtimes G \cong (F \rtimes F)/J$ , where  $J = \langle x \rtimes y \mid x, y \in F, [x, y] \in R \rangle$ . Since the centralizer of every nontrivial element of  $F$  is cyclic, we have  $F \rtimes F = F \wedge F$ . As  $H_2(F, \mathbb{Z}) = 0$ , the commutator map  $\kappa : F \wedge F \rightarrow \gamma_2(F)$  is an isomorphism. From here we conclude that  $G \rtimes G \cong \gamma_2(F)/\langle K(F) \cap R \rangle$ .

Let  $(E, \pi, A)$  be a stem extension of  $G = F/R$  producing  $\tilde{B}_0(G)$ . We have that  $A \cong \tilde{B}_0(G)$ . Let  $\{x_1, \dots, x_n\}$  be the set of free generators of  $F$ . For every  $1 \leq i \leq n$  choose  $e_i \in E$  such that  $\pi(e_i) = x_i R$ . As  $A \leq Z(E) \cap \gamma_2(E)$ , we conclude that  $A$  is contained in the Frattini subgroup  $\text{Frat}(E)$  of  $E$ , cf. [15, Satz III.3.12]. Thus  $e_1, \dots, e_n$  generate  $E$ . From here it follows that there is an epimorphism  $\sigma : F \rightarrow E$  such that  $\sigma(x_i) = e_i$  for all  $i = 1, \dots, n$ . Denote  $C = \ker \sigma$ . It is straightforward to see that  $C \leq R$ . Since  $\pi(\sigma(x)) = xR$  for every  $x \in F$ , we have that  $\sigma(R) = A$  and  $\sigma^{-1}(A) = R$ . From here we obtain  $[R, F] \leq C$ . We claim that  $\sigma(R \cap \gamma_2(F)) = A$ . For, if  $a \in A = A \cap \gamma_2(E) = \sigma(R) \cap \sigma(\gamma_2(F))$ , then we can write  $a = \sigma(r) = \sigma(\omega)$  for some  $r \in R$  and  $\omega \in \gamma_2(F)$ . It follows that  $\omega r^{-1} \in C \leq R$ , hence  $\omega \in R \cap \gamma_2(F)$ , as required. If  $\bar{\sigma}$  is the restriction of  $\sigma$  to  $R \cap \gamma_2(F)$ , then  $\ker \bar{\sigma} = C \cap \gamma_2(F)$ . Therefore we have  $(R \cap \gamma_2(F))/(C \cap \gamma_2(F)) \cong A \cong (R \cap \gamma_2(F))/\langle R \cap K(F) \rangle$ . This in particular shows that  $|C \cap \gamma_2(F) : [R, F]| = |\langle R \cap K(F) \rangle : [R, F]|$ . From here we obtain  $|\gamma_2(E)| = |\gamma_2(F) : C \cap \gamma_2(F)| = |\gamma_2(F) : [R, F]|/|C \cap \gamma_2(F) : [R, F]| = |\gamma_2(F) : [R, F]|/|\langle R \cap K(F) \rangle : [R, F]| = |\gamma_2(F) : \langle R \cap K(F) \rangle| = |G \wr G|$ .

It remains to construct a stem extension  $(E, \pi, A)$  of  $G = F/R$  producing  $\tilde{B}_0(G)$  such that  $\gamma_2(E) \cong G \wr G$ . Denote  $B = (R \cap \gamma_2(F))/\langle R \cap K(F) \rangle$  and  $T = R/\langle R \cap K(F) \rangle$ . Then  $T/B \cong R/(R \cap \gamma_2(F))$  is free abelian, hence  $B$  is complemented in  $T$ . Denote its complement by  $\bar{C} = C/\langle R \cap K(F) \rangle$ , and put  $E = F/C$ ,  $A = R/C$ . Let  $\pi : E \rightarrow G$  be the canonical epimorphism. Then  $\ker \pi = A$ . As  $[R, F] \leq \langle R \cap K(F) \rangle \leq C$ , it follows that  $(E, \pi, A)$  is a central extension of  $G$ . We have  $A \cong T/\bar{C} = B\bar{C}/\bar{C} \cong C(R \cap \gamma_2(F))/C$ , therefore  $A \leq \gamma_2(E)$ . This shows that  $(E, \pi, A)$  is a stem extension of  $G$ . As  $\gamma_2(E) \cong \gamma_2(F)/(C \cap \gamma_2(F)) = \gamma_2(F)/(C \cap (R \cap \gamma_2(F))) = \gamma_2(F)/\langle R \cap K(F) \rangle \cong G \wr G$ , the assertion is proved.  $\square$

If a group  $G$  is perfect, then  $G \wr G$  is the universal central extension of  $G$ . This can be deduced readily, cf. [22, Theorem 5.7]. A similar description can be obtained for  $G \wr G$ . We say that a central extension  $(E, \pi, A)$  of a group  $G$  is *commutativity-preserving* (CP) if commuting elements of  $G$  lift to commuting elements in  $E$ . A CP extension  $(U, \phi, A)$  of a group  $G$  is said to be *CP-universal* if for every CP extension  $(E, \psi, B)$  of  $G$  there exists a homomorphism  $\chi : U \rightarrow E$  that factors through  $G$ , i.e.,  $\psi\chi = \phi$ . It is straightforward to see that a group  $G$  admits, up to isomorphism, at most one CP-universal central extension.

The following results have their direct counterparts in the theory of universal central extensions. The proofs follow along the lines of those of [22, Chapter 5].

**Proposition 4.3.** *A CP extension  $(U, \phi, A)$  of a group  $G$  is CP-universal if and only if  $U$  is perfect, and every CP extension of  $U$  splits.*

*Proof.* Assume first that  $U$  is perfect, and that every CP extension of  $U$  splits. Let  $(E, \psi, B)$  be an arbitrary CP extension of  $G$ . Form  $U \times_G E = \{(u, e) \in U \times E \mid \phi(u) = \psi(e)\}$ , and let  $\pi : U \times_G E \rightarrow U$  be the projection to the first factor. Then  $(U \times_G E, \pi, \ker \pi)$  is a central extension of  $U$ , obviously a CP one. Thus it splits and therefore the section  $\sigma : U \rightarrow U \times_G E$  induces a homomorphism  $\chi : U \rightarrow E$ . Since  $U$  is perfect,  $\chi$  is uniquely determined [22, Lemma 5.4].

Conversely, suppose that  $(U, \phi, A)$  is a CP-universal central extension of  $G$ . Then  $U$  is perfect by [22, Lemma 5.5]. Let  $(X, \psi, B)$  be a CP extension of  $U$ . Then  $(X, \phi\psi, \ker \phi\psi)$  is a central extension of  $G$ . Take  $x, y \in G$  with  $[x, y] = 1$ . Since the extension  $(U, \phi, A)$  is CP,  $x$  and  $y$  have commuting lifts  $x', y' \in U$  with respect to  $\phi$ . The central extension  $(X, \psi, B)$  of  $U$  is also CP, hence  $x'$  and  $y'$  have commuting lifts  $x'', y'' \in X$  with respect to  $\psi$ . This shows that  $(X, \phi\psi, \ker \phi\psi)$  is a CP extension of  $G$ . By the assumption, there exists a homomorphism  $\chi : U \rightarrow X$  that factors through  $G$ . We have that  $\psi\chi$  is the identity map, hence the extension  $(X, \psi, B)$  of  $G$  splits.  $\square$

**Proposition 4.4.** *A group  $G$  admits a CP-universal central extension if and only if it is perfect. In the latter case,  $(G \wr G, \tilde{\kappa}, \tilde{B}_0(G))$  is the CP-universal central extension of  $G$ .*

*Proof.* Let  $G$  be a perfect group. Suppose  $G$  is given by the free presentation  $G = F/R$ , and denote  $K = \langle K(F) \cap R \rangle$ . We have a canonical surjection  $\rho : F/K \rightarrow F/R$ , and  $\ker \rho = R/K$  is central  $F/K$ . By [22, Lemma 5.6], the group  $\gamma_2(F)/K$ , together with the appropriate restriction of  $\rho$ , is a perfect central extension of  $\gamma_2(G) = G$ . Let  $x$  and  $y$  be commuting elements of  $G$ . Then there exist  $f_1, f_2 \in \gamma_2(F)$  such that  $x = f_1R$ ,  $y = f_2R$ , and  $[f_1, f_2] \in K(F) \cap R \subseteq K$ . This shows that the above central extension of  $G$  is CP. We claim that it is also CP-universal. Let  $(E, \psi, A)$  be another CP extension of  $G$ . As  $F$  is free, there exists a homomorphism  $\tau : F \rightarrow X$  such that  $\psi\tau = \rho$ . Take an arbitrary  $[f_1, f_2] \in K(F) \cap R$ , where  $f_1, f_2 \in F$ . Since  $\rho(f_1)$  and  $\rho(f_2)$  commute, there exist commuting lifts  $e_1, e_2 \in E$  of these with respect to  $\psi$ . We can write  $\tau(f_i) = e_i z_i$  for some  $z_i \in A \leq Z(E)$ ,  $i = 1, 2$ . Then  $\tau([f_1, f_2]) = [e_1 z_1, e_2 z_2] = 1$ , hence  $\tau$  induces a homomorphism  $\chi : F/K \rightarrow E$ . The restriction of  $\chi$  to  $\gamma_2(F)/K$  gives the required map. The second statement follows from the proof of Theorem 4.2.

The converse is obvious.  $\square$

## 5. NILPOTENT GROUPS OF CLASS 2

The first examples of finite  $p$ -groups  $G$  with  $B_0(G) \neq 0$  were found within the groups that are nilpotent of class 2, cf. [28, 3]. In this section we find a new description of  $B_0(G)$  for an arbitrary group  $G$  of class 2. This is achieved via the group  $G \wr G$ .

Let the group  $G$  be nilpotent of class 2 and consider  $G \wr G$ . As  $\gamma_2(G) \leq Z(G)$ , it follows that  $[x, y] \wr z = 1$  for all  $x, y, z \in G$ . In particular,  $G \wr G$  is an abelian group; in fact, it is easy to see that even the group  $G \wedge G$  is abelian. It also follows that

$$z(x \wedge y) = {}^z x \wedge {}^z y = [z, x]x \wedge {}^z y = x \wedge [z, y]y = x \wedge y,$$

therefore  $G$  acts trivially on  $G \wedge G$ . Thus the defining relations (4.0.1) of  $G \wr G$  show that the mapping  $G \times G \rightarrow G \wedge G$  defined by  $(x, y) \mapsto x \wedge y$  is bilinear. By the above argument, this map induces a well defined bilinear mapping  $G^{\text{ab}} \times G^{\text{ab}} \rightarrow G \wedge G$  given by  $(\bar{x}, \bar{y}) \mapsto x \wedge y$ , where  $\bar{x} = x\gamma_2(G)$  and  $\bar{y} = y\gamma_2(G)$ . This in turn induces a surjective group homomorphism  $\Psi : G^{\text{ab}} \wedge G^{\text{ab}} \rightarrow G \wedge G$  given by  $\bar{x} \wedge \bar{y} \mapsto x \wedge y$ . Similarly, there is a well defined commutator map  $G^{\text{ab}} \times G^{\text{ab}} \rightarrow \gamma_2(G)$  defined by  $(\bar{x}, \bar{y}) \mapsto [x, y]$ . Since  $G$  is of class 2, the latter mapping is also bilinear, hence it induces a surjective homomorphism  $\Phi : G^{\text{ab}} \wedge G^{\text{ab}} \rightarrow \gamma_2(G)$ . We have that  $\Phi = \tilde{\kappa}\Psi$ .

**Proposition 5.1.** *Let  $G$  be a group of class 2. Then  $\tilde{B}_0(G)$  is isomorphic to  $\ker \Phi / \ker \Psi$ .*

*Proof.* Clearly  $\ker \Psi \leq \ker \Phi$ . Let  $\tau$  be the restriction of  $\Psi$  to  $\ker \Phi$ . Take any  $k \in \ker \tilde{\kappa}$ . There exists  $t \in G^{\text{ab}} \wedge G^{\text{ab}}$  such that  $\Psi(t) = k$ . Then  $\Phi(t) = 0$ , hence  $\tau$  maps  $\ker \Phi$  onto  $\ker \tilde{\kappa}$ . Besides,  $\ker \tau = \ker \Psi$ , thus  $\ker \Phi / \ker \Psi \cong \ker \tilde{\kappa} \cong \tilde{B}_0(G)$ .  $\square$

*Remark 5.2.* We also have group homomorphisms  $\Psi_1 : G^{\text{ab}} \otimes G^{\text{ab}} \rightarrow G \wedge G$  and  $\Phi_1 : G^{\text{ab}} \otimes G^{\text{ab}} \rightarrow \gamma_2(G)$ , defined similarly as above. It can be shown that  $\tilde{B}_0(G) \cong \ker \Phi_1 / \ker \Psi_1$ .

Bogomolov [3] found a rather detailed description of  $B_0(G)$  when  $G$  is a  $p$ -group of class 2 such that  $G^{\text{ab}}$  is an elementary abelian  $p$ -group. Here we propose an alternative approach via the Blackburn-Evens theory [2]. First note that both  $\gamma_2(G)$  and  $G \wedge G$  are elementary abelian  $p$ -groups. Denote  $V = G^{\text{ab}}$  and  $W = \gamma_2(G)$ . We can consider  $V$  and  $W$  as vector spaces over  $\mathbb{F}_p$ . For  $v_1, v_2 \in V$  denote  $(v_1, v_2) =$

$[x_1, x_2]$  where  $v_i = x_i \gamma_2(G)$ . This gives us a bilinear map  $V \times V \rightarrow W$ . Let  $X_1$  be the subspace of  $V \otimes W$  spanned by all  $v_1 \otimes (v_2, v_3) + v_2 \otimes (v_3, v_1) + v_3 \otimes (v_1, v_2)$ , where  $v_i \in V$ . Furthermore, define the map  $f : V \rightarrow W$  by  $f(g\gamma_2(G)) = g^p$ , and let  $X_2$  be the subspace of  $V \otimes W$  spanned by all  $v \otimes f(v)$ , where  $v \in V$ . Put  $X = X_1 + X_2$ . Straightforward verification, cf. [2], shows that the map  $\sigma : V \wedge V \rightarrow (V \otimes W)/X$  given by  $\sigma(v_1 \wedge v_2) = v_1 \otimes f(v_2) + \binom{p}{2} v_2 \otimes (v_1, v_2) + X$  is well defined and  $\mathbb{F}_p$ -linear. As both  $V \wedge V$  and  $(V \otimes W)/X$  are elementary abelian  $p$ -groups, there exists an elementary abelian  $p$ -group  $M^*$  with  $N \leq M^*$  such that

$$(5.2.1) \quad N \cong (V \otimes W)/X \quad \text{and} \quad M^*/N \cong V \wedge V.$$

**Theorem 5.3.** *Let  $G$  be a finite group of class 2 such that  $G^{\text{ab}}$  is an elementary abelian  $p$ -group. Under the isomorphisms given by (5.2.1), let  $M/N$  correspond to  $\ker \Phi$  in  $M^*/N$ , and  $M_0/N$  correspond to  $\ker \Psi$  in  $M^*/N$ . Then  $\tilde{B}_0(G) \cong M/M_0$  and  $M_0(G) \cong M_0$ .*

*Proof.* By a result of Blackburn and Evens [2],  $M \cong M(G)$ . Proposition 5.1 implies that  $\tilde{B}_0(G) \cong \ker \Phi / \ker \Psi \cong M/M_0$ . Since both  $M$  and  $M_0$  are elementary abelian, it follows from Theorem 3.2 that  $M_0(G) \cong M_0$ . This concludes the proof.  $\square$

## 6. SPLIT EXTENSIONS AND FROBENIUS GROUPS

In this section all the groups are finite. Let  $G = N \rtimes Q$  be a split extension of the group  $N$  by  $Q$ . Then the Schur multiplier of  $G$  can be described by a result of Tahara [33], see also [17, p. 28]. We have that  $H^2(G, \mathbb{Q}/\mathbb{Z})$  is naturally isomorphic to  $H^2(Q, \mathbb{Q}/\mathbb{Z}) \oplus \bar{H}^2(G, \mathbb{Q}/\mathbb{Z})$ , where  $\bar{H}^2(G, \mathbb{Q}/\mathbb{Z}) = \ker \text{res}_Q^G$ . Moreover,  $\bar{H}^2(G, \mathbb{Q}/\mathbb{Z})$  fits into the following exact sequence:

$$(6.0.1) \quad 0 \rightarrow H^1(Q, H^1(N, \mathbb{Q}/\mathbb{Z})) \rightarrow \bar{H}^2(G, \mathbb{Q}/\mathbb{Z}) \rightarrow H^2(N, \mathbb{Q}/\mathbb{Z})^Q \rightarrow H^2(Q, H^1(N, \mathbb{Q}/\mathbb{Z})).$$

A description in terms of the nonabelian exterior products is obtained as follows. The commutator map  $G \wedge N \rightarrow [N, G]$  is a homomorphism of groups. Denote its kernel by  $M(G, N)$ . The group  $M(G, N)$  is said to be the *Schur multiplier of the pair*  $(G, N)$ . Ellis [12] proved that  $M(G) \cong M(G, N) \oplus M(Q)$ , and  $M(G, N) \cong \ker(M(G) \rightarrow M(Q))$ . Here  $M(G, N)$  is embedded into  $M(G)$  via the restriction  $\iota_1$  of the natural homomorphism  $G \wedge N \rightarrow G \wedge G$ , and the embedding  $\iota_2 : M(Q) \hookrightarrow M(G)$  is induced by the split surjection  $G \twoheadrightarrow Q$ .

Our aim is to describe  $\tilde{B}_0(G)$  in the case when  $G$  is a split extension of  $N$  by  $Q$ . At first we define a subgroup  $\bar{M}_0(G, N)$  of  $G \wedge N$  by

$$\bar{M}_0(G, N) = \langle (a \wedge m)^{-1} (b \wedge n) (n \wedge m) \mid a, b \in G, m, n \in N, [a, b] = 1, {}^b n m = {}^a m n \rangle.$$

It is straightforward to verify that  $M_0(G, N) \leq \bar{M}_0(G, N) \leq M(G, N)$ .

**Theorem 6.1.** *Let  $G = N \rtimes Q$ . Then  $\tilde{B}_0(G) \cong M(G, N)/\bar{M}_0(G, N) \oplus \tilde{B}_0(Q)$ .*

*Proof.* Let  $x, y \in G$  commute. We can write  $x = n_1^{-1} q_1$  and  $y = n_2^{-1} q_2$  for some  $n_1, n_2 \in N$  and  $q_1, q_2 \in Q$ . From  $[x, y] = 1$  we obtain  $n_1^{-1} [q_1, n_2] \cdot n_1^{-1} n_2^{-1} [q_1, q_2] \cdot [n_1^{-1}, y] = 1$ . As  $N \cap Q = 1$ , we conclude that  $[q_1, q_2] = 1$ . Therefore  $q_1 \wedge q_2$  as an element of  $G \wedge G$  belongs to  $\iota_2 M_0(Q)$ , hence it is central in  $G \wedge G$  and  $G$  acts trivially upon it. Now we have

$$\begin{aligned} x \wedge y &= n_1^{-1} (q_1 \wedge n_2^{-1}) \cdot n_1^{-1} n_2^{-1} (q_1 \wedge q_2) \cdot (n_1^{-1} \wedge n_2^{-1}) \cdot n_2^{-1} (n_1^{-1} \wedge q_2) \\ &= n_1^{-1} n_2^{-1} \left( n_2 (q_1 \wedge n_2^{-1}) \cdot n_1 n_2 (n_1^{-1} \wedge n_2^{-1}) \cdot [n_2, n_1] n_1 (n_1^{-1} \wedge q_2) \right) (q_1 \wedge q_2) \\ &= n_1^{-1} n_2^{-1} \left( (q_1 \wedge n_2)^{-1} (q_2 \wedge n_1) (n_1 \wedge n_2) \right) (q_1 \wedge q_2). \end{aligned}$$

From  $[x, y] = [q_1, q_2] = 1$  we obtain that  $[n_2, q_1][q_2, n_1][n_1, n_2] = 1$ , which is equivalent to  ${}^{q_2}n_1n_2 = {}^{q_1}n_2n_1$ . It follows from here that  $(q_1 \wedge n_2)^{-1}(q_2 \wedge n_1)(n_1 \wedge n_2) \in \iota_1 \bar{M}_0(G, N)$ . This shows that  $M_0(G) \leq \iota_1 \bar{M}_0(G, N) \oplus \iota_2 M_0(Q)$ . Conversely, it is clear that  $\iota_2 M_0(Q) \leq M_0(G)$ . Take any generator  $\omega = (a \wedge m)^{-1}(b \wedge n)(n \wedge m)$  of  $\bar{M}_0(G, N)$ . Then we have that  $[a, b] = 1$  and  ${}^b n m = {}^a m n$ . The above calculation shows that  ${}^{n^{-1}m^{-1}}\omega = (n^{-1}a \wedge m^{-1}b)(b \wedge a)$ . By our assumptions we have  $[n^{-1}a, m^{-1}b] = 1$ , therefore  $\omega \in M_0(G)$ . From here we can finally conclude that  $M_0(G) = \iota_1 \bar{M}_0(G, N) \oplus \iota_2 M_0(Q)$ , and this proves the assertion.  $\square$

The structure of  $\tilde{B}_0(G)$  can further be refined when  $G$  is a Frobenius group. A *Frobenius group* [15, p. 496] is a transitive permutation group such that no non-trivial element fixes more than one point and some non-trivial element fixes a point. The subgroup  $Q$  of a Frobenius group  $G$  fixing a point is called the *Frobenius complement*. By a theorem of Frobenius [15, p. 496], the set

$$N = G \setminus \bigcup_{g \in G} {}^g(Q \setminus \{1\})$$

is a normal subgroup in  $G$  called the *Frobenius kernel*  $N$ . We have that  $G = N \rtimes Q$ , and  $Q$  acts fixed-point-freely upon  $N$ . We have that  $Q \cap {}^g Q = 1$  for every  $g \in G \setminus Q$ , and so if  $\{g_1, \dots, g_r\}$  is a left transversal of  $Q$  in  $G$  then we have a *Frobenius partition*

$$(6.1.1) \quad G = N \dot{\cup} {}^{g_1}Q \dot{\cup} \dots \dot{\cup} {}^{g_r}Q,$$

where the word ‘partition’ means that the intersection of two different components is 1.

At first we describe the Schur multiplier of a Frobenius group by refining the above mentioned result of Tahara.

**Proposition 6.2.** *Let  $G$  be a Frobenius group with Frobenius kernel  $N$  and complement  $Q$ . Then  $H^2(G, \mathbb{Q}/\mathbb{Z}) \cong H^2(N, \mathbb{Q}/\mathbb{Z})^Q \cong M(G, N)$ .*

*Proof.* By Tahara’s result we have  $H^2(G, \mathbb{Q}/\mathbb{Z}) \cong H^2(Q, \mathbb{Q}/\mathbb{Z}) \oplus \bar{H}^2(G, \mathbb{Q}/\mathbb{Z})$ . The Sylow  $p$ -subgroups of  $Q$  are cyclic if  $p$  is odd, and either cyclic or generalized quaternion groups if  $p = 2$  [15, Hauptsatz V.8.7]. Thus  $H^2(P, \mathbb{Q}/\mathbb{Z}) = 0$  for every Sylow  $p$ -subgroup  $P$  of  $Q$  and every prime  $p$  dividing the order of  $Q$ . It follows from here that  $H^2(Q, \mathbb{Q}/\mathbb{Z}) = 0$ . It remains to show that  $\bar{H}^2(G, \mathbb{Q}/\mathbb{Z}) \cong H^2(N, \mathbb{Q}/\mathbb{Z})^Q$ . By [15, Satz V.8.3] we have  $\gcd(|N|, |Q|) = 1$ , which clearly implies  $H^i((Q, H^1(N, \mathbb{Q}/\mathbb{Z}))) = 1$  for all  $i \geq 1$ , hence the exact sequence (6.0.1) gives the isomorphism  $H^2(G, \mathbb{Q}/\mathbb{Z}) \cong H^2(N, \mathbb{Q}/\mathbb{Z})^Q$ . The fact that the latter is isomorphic to  $M(G, N)$  follows from the above mentioned result of Ellis.  $\square$

Moving on to  $\tilde{B}_0(G)$ , where  $G$  is a Frobenius group, we first need to describe the structure of commuting pairs in  $G$ . In the Frobenius case, these are particularly well behaved, as the following result shows.

**Lemma 6.3.** *Let  $G$  be a Frobenius group with the Frobenius kernel  $N$  and complement  $Q$ . Let  $x, y \in G$  commute. Then either  $x, y \in N$  or there exists  $g \in G$  such that both  $x$  and  $y$  belong to  ${}^g Q$ .*

*Proof.* Let  $G$  have a Frobenius partition as given by (6.1.1). Suppose that  $[x, y] = 1$  for  $x, y \in G \setminus \{1\}$ . We may further suppose that at least one of these elements does not belong to  $N$ . Assume first that  $x \in N$  and  $y \notin N$ . Without loss of generality we can write  $y = {}^{g_1}q$  for some  $q \in Q$ . We have  ${}^{xg_1}q = {}^{g_1}q$ , therefore  ${}^{g_1^{-1}}xq = q$ . This can be rewritten as  ${}^q \left( {}^{g_1^{-1}}x \right) = {}^{g_1^{-1}}x$ . Since  $Q$  acts fixed-point-freely on  $N$ , we conclude that  $q = 1$  or  $x = 1$ , a contradiction.

Assume now that  $x$  and  $y$  belong to different conjugates of  $Q$ . Without loss of generality we may assume that  $x \in Q$  and  $y \in {}^{g_1}Q$  where  $g_1 \notin Q$ . We can write  $y = {}^{g_1}q$ , where  $q \in Q$  and  $g_1 = f_1 q_1$  with  $f_1 \in F \setminus \{1\}$  and  $q_1 \in Q$ . Denote  $\tilde{q} = {}^{q_1}q$ . From  ${}^x y = y$  we conclude that  ${}^{f_1^{-1}x f_1} \tilde{q} = \tilde{q} \in Q \cap {}^{f_1^{-1}x f_1}Q$ . As  $\tilde{q} \neq 0$ , we obtain that  ${}^{f_1^{-1}x f_1} \in Q$ , hence  $x \in {}^{f_1}Q$ . But  $Q \cap {}^{f_1}Q = 1$ , and this is contrary to the assumption that  $x \neq 1$ . This concludes the proof.  $\square$

**Corollary 6.4.** *Let  $G$  be a Frobenius group with the Frobenius kernel  $N$ . Then*

$$\tilde{B}_0(G) \cong \frac{M(G, N)}{\text{im}(M_0(N) \rightarrow M(G, N))}.$$

*Proof.* Denote  $N_0 = \text{im}(M_0(N) \rightarrow M(G, N))$ . Let  $x, y \in G$  and suppose that  $[x, y] = 1$ . By Lemma 6.3 we either have that  $x, y \in N$  or there exists  $g \in G$  such that  $x = {}^g q_1$  and  $y = {}^g q_2$  for some  $q_1, q_2 \in Q$ . We clearly have that  $[q_1, q_2] = 1$ , hence  $x \wedge y = {}^g q_1 \wedge {}^g q_2 = {}^g(q_1 \wedge q_2) = q_1 \wedge q_2$ . This shows that  $M_0(G) = \langle x \wedge y \mid [x, y] = 1, \text{ either } (x, y) \in N \times N \text{ or } (x, y) \in Q \times Q \rangle$ . In view of the above notations we can thus write  $M_0(G) = \iota_1 N_0 \oplus \iota_2 M_0(Q)$ . As  $H^2(Q, \mathbb{Q}/\mathbb{Z}) = 0$ , we have that  $M_0(Q) = 0$ , and hence the result.  $\square$

**Corollary 6.5.** *Let  $G$  be a Frobenius group with the Frobenius kernel  $N$ . Then*

$$B_0(G) = \bigcap_{A \in \mathcal{C}} \ker \text{res}_A^G,$$

where  $\mathcal{C}$  is the family of all bicyclic subgroups of  $N$ .

*Proof.* Denote  $B_0 = \bigcap_{A \in \mathcal{C}} \ker \text{res}_A^G$ . By a result of Bogomolov [3] we have that  $B_0(G) = \bigcap_{A \in \mathcal{B}} \ker \text{res}_A^G$ , where  $\mathcal{B}$  is the collection of all bicyclic subgroups of  $G$ , hence  $B_0(G) \leq B_0$ . Now let  $\gamma \in B_0$ . Fix an arbitrary  $B = \langle x, y \rangle \in \mathcal{B}$ . If  $x, y \in N$ , then  $B \in \mathcal{C}$ , and thus  $\text{res}_B^G \gamma = 0$ . Otherwise, Lemma 6.3 implies that there exists  $g \in G$  such that  $x = {}^g q_1$  and  $y = {}^g q_2$  for some  $q_1, q_2 \in Q$ . Clearly we have  $[q_1, q_2] = 1$ . As  $H^2({}^g Q, \mathbb{Q}/\mathbb{Z}) = 0$ , we have  $H^2(G, \mathbb{Q}/\mathbb{Z}) = \ker \text{res}_g^G \leq \ker \text{res}_B^G$ , hence we again have  $\text{res}_B^G \gamma = 0$ . We conclude that  $\gamma \in B_0(G)$ .  $\square$

**Corollary 6.6.** *Let  $G$  be a Frobenius group with abelian Frobenius kernel. Then  $B_0(G) = 0$ .*

*Proof.* Let  $N$  be the Frobenius kernel of  $G$  and  $Q$  a complement of  $N$  in  $G$ . As  $N$  is abelian, application of Corollary 6.5 gives  $B_0(G) = \ker \text{res}_N^G$ . Thus it suffices to show that the map  $\text{res}_N^G$  is injective. Let  $\text{cor}_N^G : H^2(N, \mathbb{Q}/\mathbb{Z}) \rightarrow H^2(G, \mathbb{Q}/\mathbb{Z})$  be the cohomological corestriction map. Let  $p$  be a prime dividing  $|N|$ , and denote the restriction of the map  $\text{res}_N^G$  to the  $p$ -part  $H^2(G, \mathbb{Q}/\mathbb{Z})_p$  of  $H^2(G, \mathbb{Q}/\mathbb{Z})$  by  $\text{res}_N^G(p)$ . Similarly, let  $\text{cor}_N^G(p)$  be the restriction of  $\text{cor}_N^G$  to  $H^2(N, \mathbb{Q}/\mathbb{Z})_p$ . Then  $\text{cor}_N^G(p) \text{res}_N^G(p) : H^2(G, \mathbb{Q}/\mathbb{Z})_p \rightarrow H^2(G, \mathbb{Q}/\mathbb{Z})_p$  is multiplication by  $n = |G : N| = |Q|$ . As  $p$  is coprime to  $n$ , it follows that  $\text{res}_N^G(p)$  is injective for every  $p$  dividing  $|N|$ . Therefore  $\text{res}_N^G$  is injective, as required.  $\square$

## 7. THE FUNCTOR $\tilde{B}_0$ IN K-THEORY

In this section, the role of the functor  $\tilde{B}_0$  within K-theory is outlined. We first briefly recall some of the basic notions of K-theory. For unexplained notations and further account we refer to Milnor's book [22]. Throughout this section let  $\Lambda$  be a ring with 1. The group  $\text{GL}(\Lambda)$  is the direct limit of the chain  $\text{GL}(1, \Lambda) \subset \text{GL}(2, \Lambda) \subset \dots$ , where  $\text{GL}(n, \Lambda)$  is embedded in  $\text{GL}(n+1, \Lambda)$  via  $A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$ . Denote by  $E(\Lambda)$  the subgroup of  $\text{GL}(\Lambda)$  generated by all elementary matrices, and let  $\text{St}(\Lambda)$  be the Steinberg group. Then the  $K_1$  and  $K_2$  functors are given by  $K_1 \Lambda = \text{GL}(\Lambda)/E(\Lambda)$

and  $K_2 \Lambda = \ker(\Phi : \text{St}(\Lambda) \rightarrow E(\Lambda))$ , respectively. It is known that  $K_2 \Lambda$  is precisely the center of  $\text{St}(\Lambda)$ ,  $K_2 \Lambda \cong H_2(E(\Lambda), \mathbb{Z})$ , and the sequence

$$1 \longrightarrow K_2 \Lambda \longrightarrow \text{St}(\Lambda) \longrightarrow \text{GL}(\Lambda) \longrightarrow K_1 \Lambda \longrightarrow 1$$

is exact.

The fact that  $K_2 \Lambda$  can be identified with  $H_2(E(\Lambda), \mathbb{Z})$  suggests the following definition. For a ring  $\Lambda$  set  $\tilde{B}_0 \Lambda = \tilde{B}_0(E(\Lambda))$ . This clearly defines a covariant functor from **Ring** to **Ab**. The group  $\tilde{B}_0 \Lambda$  fits into the exact sequence

$$1 \longrightarrow \tilde{B}_0 \Lambda \longrightarrow E(\Lambda) \wedge E(\Lambda) \longrightarrow E(\Lambda) \longrightarrow 1.$$

Thus  $\tilde{B}_0 \Lambda$  can be considered a measure of the extent to which relations among commutators in  $\text{GL}(\Lambda)$  fail to be consequences of ‘universal’ relations of  $E(\Lambda) \wedge E(\Lambda)$ . Another description of  $\tilde{B}_0 \Lambda$  can be obtained via the Steinberg group. Denote  $M_0 \Lambda = (K(\text{St}(\Lambda)) \cap K_2 \Lambda)$ . Then we have the following result.

**Theorem 7.1.** *Let  $\Lambda$  be a ring. Then  $E(\Lambda) \wedge E(\Lambda)$  is naturally isomorphic to  $\text{St}(\Lambda)/M_0 \Lambda$ , and  $\tilde{B}_0 \Lambda \cong K_2 \Lambda/M_0 \Lambda$ .*

*Proof.* The group  $\text{St}(\Lambda)$  is the universal central extension of  $E(\Lambda)$  [22, Theorem 5.10]. Since  $\text{St}(\Lambda)$  is perfect, it follows from [21] that  $\text{St}(\Lambda) \cong E(\Lambda) \wedge E(\Lambda)$ . The isomorphism  $\psi : E(\Lambda) \wedge E(\Lambda) \rightarrow \text{St}(\Lambda)$  can be chosen so that we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 1 & \longrightarrow & M(E(\Lambda)) & \longrightarrow & E(\Lambda) \wedge E(\Lambda) & \xrightarrow{\kappa} & E(\Lambda) \longrightarrow 1 \\ & & \downarrow \psi|_{M(E(\Lambda))} \cong & & \downarrow \psi \cong & & \parallel \\ 1 & \longrightarrow & K_2 \Lambda & \longrightarrow & \text{St}(\Lambda) & \xrightarrow{\Phi} & E(\Lambda) \longrightarrow 1 \end{array}$$

From here we get that

$$E(\Lambda) \wedge E(\Lambda) = (E(\Lambda) \wedge E(\Lambda))/M_0(E(\Lambda)) \cong \text{St}(\Lambda)/\psi(M_0(E(\Lambda))).$$

As  $\psi(M_0(E(\Lambda))) = \langle [x, y] \mid x, y \in \text{St}(\Lambda), [\Phi(x), \Phi(y)] = 1 \rangle = \langle [x, y] \mid x, y \in \text{St}(\Lambda), [x, y] \in K_2 \Lambda \rangle = M_0 \Lambda$ , we get the result.  $\square$

Theorem 7.1 thus shows that  $\tilde{B}_0 \Lambda$  is the obstruction to  $K_2 \Lambda$  being generated by commutators. Alternatively, let  $A, B \in E(\Lambda)$  commute, and choose  $a, b \in \text{St}(\Lambda)$  such that  $A = \Phi(a)$  and  $B = \Phi(b)$ . Define  $A \star B = [a, b] \in K_2 \Lambda$  to be the *Milnor element* induced by  $A$  and  $B$ , cf. [22, p. 63]. The following is then straightforward.

**Proposition 7.2.** *Let  $\Lambda$  be a ring. Then  $M_0 \Lambda = \langle A \star B \mid A, B \in E(\Lambda), [A, B] = 1 \rangle$ . Thus  $\tilde{B}_0 \Lambda = 0$  if and only if  $K_2 \Lambda$  is generated by Milnor’s elements.*

The question as to whether  $K_2 \Lambda$  is generated by Milnor’s elements for every ring  $\Lambda$  was posed by Bass, cf. Problem 3 of [10]. As the group  $E(\Lambda)$  is perfect, the problem is equivalent to the question whether or not every CP extension of  $E(\Lambda)$  is trivial, cf. Proposition 4.4.

Now let  $\{x_{ij}^\lambda \mid i, j \in \mathbb{N}, \lambda \in \Lambda\}$  be the standard generating set of  $\text{St}(\Lambda)$ . For  $u \in \Lambda^\times$  define  $w_{ij}(u) = x_{ij}^u x_{ji}^{-u^{-1}} x_{ij}^u$  and  $h_{ij}(u) = w_{ij}(u)w_{ij}(-1)$ . For  $u, v \in \Lambda^\times$  with  $uv = vu$  let  $\{u, v\} = [h_{ij}(u), h_{ij}(v)]$  be the *Steinberg symbol*. It is known that  $K_2 \mathbb{Z}$  is generated by the Steinberg symbol  $\{-1, -1\}$ , cf. [22, Corollary 10.2]. This, together with Theorem 7.1, implies that  $\tilde{B}_0 \mathbb{Z} = 0$ . Similarly, we have the following.

**Corollary 7.3.** *Let  $\Lambda$  be a commutative semilocal ring. Then  $\tilde{B}_0 \Lambda = 0$ .*

*Proof.* By a result of Stein and Dennis [31, Theorem 2.7],  $K_2 \Lambda$  is generated by the Steinberg symbols  $\{u, v\}$ , where  $u, v \in \Lambda^\times$ , hence the result follows from Theorem 7.1.  $\square$



Our next goal is to compute  $\tilde{B}_0(\mathrm{GL}(\Lambda))$  for an arbitrary ring  $\Lambda$ . Dennis [9, Corollary 8] showed that  $H_2(\mathrm{GL}(\Lambda), \mathbb{Z}) \cong K_2 \Lambda \oplus H_2(\mathrm{GL}(\Lambda)^{\mathrm{ab}}, \mathbb{Z})$ . The drawback is that the splitting is non-canonical. Instead of that, we use a variant of the functor  $H_2$  defined by Dennis. Given a group  $G$ , let  $G \tilde{\wedge} G$  be the group generated by symbols  $x \tilde{\wedge} y$ , where  $x, y \in G$  are subject to the relations analogous to (2.0.1) and (2.0.2) in the definition of the nonabelian exterior square  $G \wedge G$  of the group  $G$ , and the relation (2.0.3) is replaced by

$$(7.3.1) \quad (x \tilde{\wedge} y)(y \tilde{\wedge} x) = 1$$

for all  $x, y \in G$ . We clearly have the commutator homomorphism  $\hat{\kappa} : G \tilde{\wedge} G \rightarrow \gamma_2(G)$  given by  $x \tilde{\wedge} y \mapsto [x, y]$ . Denote  $\tilde{H}_2(G) = \ker \hat{\kappa}$ . The latter group has a topological interpretation. Namely, it follows from [6] that  $\tilde{H}_2(G) \cong \pi_4(\Sigma^2 K(G, 1))$ , where  $K(G, 1)$  is the classifying space of  $G$ . Let  $\tilde{M}_0(G) = \langle x \tilde{\wedge} y \mid x, y \in G, [x, y] = 1 \rangle$ . Then the defining relations of  $G \tilde{\wedge} G$  imply that  $(G \tilde{\wedge} G)/\tilde{M}_0(G) \cong G \wr G$  and  $\tilde{H}_2(G)/\tilde{M}_0(G) \cong \tilde{B}_0(G)$ . If  $G$  is perfect, then  $(G \tilde{\wedge} G, \hat{\kappa}, \tilde{H}_2(G))$  is the universal central extension of  $G$ .

In our context it is crucial to note that there is a canonical split exact sequence

$$(7.3.2) \quad 1 \longrightarrow \tilde{H}_2(E(\Lambda)) \longrightarrow \tilde{H}_2(\mathrm{GL}(\Lambda)) \longrightarrow \tilde{H}_2(\mathrm{GL}(\Lambda)^{\mathrm{ab}}) \longrightarrow 1,$$

cf. [9, Theorem 7]. This facilitates the proof of the following result:

**Theorem 7.4.** *Let  $\Lambda$  be a ring. Then  $\tilde{B}_0 \Lambda$  is naturally isomorphic to  $\tilde{B}_0(\mathrm{GL}(\Lambda))$ .*

*Proof.* Let  $G = \mathrm{GL}(\Lambda)$  and  $E = E(\Lambda)$ . By [9, Theorem 7] we have that  $G \tilde{\wedge} G$  is naturally isomorphic to  $(E \tilde{\wedge} E) \times (G^{\mathrm{ab}} \tilde{\wedge} G^{\mathrm{ab}})$ . Explicitly, there is a pairing  $\star : G \times G \rightarrow E \tilde{\wedge} E \cong \mathrm{St}(\Lambda)$  that extends the Milnor pairing defined above. This was found by Grayson, see [14] for the details. It turns out [14, p. 27] that the map  $\star$  preserves the relations (2.0.1), (2.0.2) and (7.3.1), hence it induces a well defined homomorphism  $\star : G \tilde{\wedge} G \rightarrow E \tilde{\wedge} E$ . We have that  $G^{\mathrm{ab}} \tilde{\wedge} G^{\mathrm{ab}} = \tilde{H}_2(G^{\mathrm{ab}})$ , and the pairing  $\circ : G \times G \rightarrow \tilde{H}_2(G^{\mathrm{ab}})$  given by  $a \circ b = (a \oplus 1)\gamma_2(G) \wedge (1 \oplus b)\gamma_2(G)$  induces a homomorphism  $\circ : G \tilde{\wedge} G \rightarrow \tilde{H}_2^2(G^{\mathrm{ab}})$ . It can be proved [9] that  $x \tilde{\wedge} y = (x \star y)(x \circ y)$  for every  $x, y \in G$ . From the definition of  $\star$  it follows that if  $x$  and  $y$  commute, then  $x \star y \in K_2 \Lambda$ , and the elements  $x \circ y$  generate  $\tilde{H}_2^2(G^{\mathrm{ab}})$ . Therefore  $\tilde{M}_0(G) = \tilde{M}_0(E) \times \tilde{H}_2^2(G^{\mathrm{ab}})$ . This gives the result.  $\square$

One of the fundamental results in K-theory is that if  $\Lambda$  is a ring and  $I$  an ideal of  $\Lambda$ , then the sequence

$$(7.4.1) \quad K_2(\Lambda, I) \longrightarrow K_2 \Lambda \xrightarrow{\tau} K_2(\Lambda/I) \xrightarrow{\partial} K_1(\Lambda, I) \longrightarrow K_1 \Lambda \longrightarrow \dots$$

is exact [22, Theorem 6.2]. In the rest of the section we derive a similar sequence for  $\tilde{B}_0$ . To this end, denote  $J(\Lambda, I) = \partial(\tilde{M}_0(\Lambda/I))$  and  $T(\Lambda, I) = \tau^{-1}(\tilde{M}_0(\Lambda/I))$ . Then we have the following.

**Proposition 7.5.** *Let  $\Lambda$  be a ring and  $I$  an ideal of  $\Lambda$ . Then the sequence*

$$1 \longrightarrow \frac{T(\Lambda, I)}{\tilde{M}_0 \Lambda} \longrightarrow \tilde{B}_0 \Lambda \longrightarrow \tilde{B}_0(\Lambda/I) \longrightarrow \frac{K_1(\Lambda, I)}{J(\Lambda, I)} \longrightarrow K_1 \Lambda \longrightarrow \dots$$

*is exact.*

*Proof.* The canonical homomorphism  $\tau : K_2 \Lambda \rightarrow K_2(\Lambda/I)$  induces a homomorphism  $\tau^\sharp : \tilde{B}_0 \Lambda \rightarrow \tilde{B}_0(\Lambda/I)$ , whose kernel is precisely  $T(\Lambda, I)/\tilde{M}_0 \Lambda$ . By definition, the connecting homomorphism  $\partial : K_2(\Lambda/I) \rightarrow K_1(\Lambda, I)$  induces a natural map  $\partial^\sharp : \tilde{B}_0(\Lambda/I) \rightarrow K_1(\Lambda, I)/J(\Lambda, I)$ , and we have that  $\ker \partial^\sharp = \tilde{M}_0(\Lambda/I) \ker \partial / \tilde{M}_0(\Lambda/I) = \mathrm{im} \tau^\sharp$ . Using the fact that the sequence (7.4.1) is exact, we see that the canonical homomorphism  $\sigma : K_1(\Lambda, I) \rightarrow K_1 \Lambda$  induces a well defined homomorphism  $\sigma^\sharp :$

$K_1(\Lambda, I)/J(\Lambda, I) \rightarrow K_1 \Lambda$ . We have that  $\text{im } \sigma^\# = \text{im } \sigma$ , and  $\ker \sigma^\# = \ker \sigma/J(\Lambda, I) = \text{im } \partial/J(\Lambda, I) = \text{im } \partial^\#$ . This concludes the proof.  $\square$

## 8. COMPUTING $\tilde{B}_0(G)$

A group  $G$  is said to be *polycyclic* if it has a subnormal series  $1 = G_0 \triangleleft G_1 \triangleleft \dots \triangleleft G_n = G$  such that every factor  $G_{i+1}/G_i$  is cyclic. A finite group is polycyclic if and only if it is solvable. Computations with polycyclic groups are very efficient, since several algorithmic problems are decidable within this class [30].

Recently Eick and Nickel [11] developed efficient algorithms for computing non-abelian exterior squares and Schur multipliers of (possibly infinite) polycyclic groups. Given a polycyclic group  $G$ , one can compute its nonabelian exterior square  $G \wedge G$ , the crossed pairing  $\lambda : G \times G \rightarrow G \wedge G$  given by  $\lambda(x, y) = x \wedge y$ , and the commutator map  $\kappa : G \wedge G \rightarrow \gamma_2(G)$ . The Schur multiplier  $H_2(G, \mathbb{Z})$  is then computed as  $M(G) = \ker \kappa$ .

Let  $G$  be a finite solvable group. In order to compute  $\tilde{B}_0(G)$  it suffices to efficiently compute  $M_0(G) = \langle x \wedge y \mid x, y \in G, [x, y] = 1 \rangle$  as a subgroup of  $M(G)$ . One would have to compute the set  $\mathcal{C}_G = \{(x, y) \in G \times G \mid [x, y] = 1\}$  of all commuting pairs in  $G$  and then to compute  $M_0(G)$  as the group generated by  $\{\lambda(x, y) \mid (x, y) \in \mathcal{C}_G\}$ . It turns out that this is computationally inefficient. The first improvement is to observe that if  $(x, y) \in \mathcal{C}_G$ , then also  $({}^z x, {}^z y) \in \mathcal{C}_G$  for every  $z \in G$ . On the other hand, since  $G$  acts trivially on  $M(G)$ , we have that  ${}^z x \wedge {}^z y = {}^z(x \wedge y) = x \wedge y$ , therefore it suffices to determine the conjugacy classes  $C_1, \dots, C_k$  and choose representatives  $c_i \in C_i$ ,  $i = 1, \dots, k$ . Then  $M_0(G) = \langle c_i \wedge x \mid c_i \in C_i, x \in C_G(c_i), i = 1, \dots, k \rangle$ . This can further be improved. For  $x \in G$  consider the map  $\varphi : C_G(x) \rightarrow \ker \kappa$  given by  $y \mapsto x \wedge y$ . Let  $y, z \in C_G(x)$ . Then  $x \wedge yz = (x \wedge y)({}^y x \wedge {}^y z) = (x \wedge y)(x \wedge z)$ , as  $G$  acts trivially on  $\ker \kappa$ . Thus  $\varphi$  is a homomorphism. It follows from here that if  $\mathcal{X}_i$  is a generating set of  $C_G(c_i)$ ,  $i = 1, \dots, k$ , then

$$M_0(G) = \langle c_i \wedge x \mid c_i \in C_i, x \in \mathcal{X}_i, i = 1, \dots, k \rangle.$$

This formula enables efficient computation of  $M_0(G)$ , as it provides a reasonably small set of generators of this group. The algorithm has been implemented in GAP [13]. It allows us to compute  $\tilde{B}_0(G)$  and  $G \wedge G$  for finite solvable groups  $G$ . A file of the GAP functions and commands for computing  $\tilde{B}_0(G)$  can be found at the author's website [23].

Computer experiments reveal that there are no groups  $G$  of order 32 with  $B_0(G) \neq 0$ . This coincides with the hand calculations done by Chu, Hu, Kang and Prokhorov [8]. Next, there are nine groups  $G$  of order 64 with  $B_0(G) \neq 0$ . If we denote the  $i$ -th group in the GAP library of all groups of order  $n$  by  $G_n(i)$ , then our computations using the above algorithm show that we have  $B(G_{64}(i)) \neq 0$  for  $i \in \{149, 150, 151, 170, 171, 172, 177, 178, 182\}$ . In fact, in all these cases  $B_0(G_{64}(i))$  is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$ . This confirms the calculations of Chu, Hu, Kang, and Kunyavskii [7, Theorem 10.8].

Bogomolov [3, Lemma 4.11] stated that if  $G$  is a group with  $G^{\text{ab}} \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$  and  $B_0(G) \neq 0$ , then  $p > 3$  and  $|G|$  has to be at least  $p^7$ . His methods also imply that if  $G$  is a  $p$ -group with  $B_0(G) \neq 0$ , then  $|G| \geq p^6$ , cf. [4, Corollary 2.11]. On the other hand, our computations show that if  $i \in \{28, 29, 30\}$ , then  $G_{243}(i)^{\text{ab}} \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$  and  $B_0(G_{243}(i)) \cong \mathbb{Z}/3\mathbb{Z}$ . These can be double-checked by hand calculations using the methods of [7], and thus contradict both of the above Bogomolov's claims. We only sketch here the relevant computations with the group  $G_{243}(28)$ .

*Example 8.1.* Denote  $G = G_{243}(28)$ . This group has the following polycyclic presentation:

$$G = \langle g_1, g_2, g_3, g_4, g_5 \mid g_1^3 = 1, g_2^3 = g_4^2, g_3^3 = g_5^2, g_4^3 = 1, g_5^3 = 1, [g_2, g_1] = g_3, \\ [g_3, g_1] = g_4, [g_3, g_2] = g_5, [g_4, g_1] = g_5, [g_i, g_j] = 1 \text{ for other } i > j \rangle.$$

Computations with GAP show that  $G \wedge G$  is isomorphic to  $G_{243}(34)$ , and is generated by the set  $\{g_2 \wedge g_1, g_3 \wedge g_1, g_3 \wedge g_2, g_4 \wedge g_1\}$ . Denote  $w = (g_2 \wedge g_3)(g_4 \wedge g_1)$ . We have that  $|w| = 9$ , and since  $[g_2, g_3][g_4, g_1] = 1$ , it follows that  $w \in M(G)$ . Further inspection of  $G \wedge G$  reveals that  $M(G) = \langle w \rangle$  and  $M_0(G) \cong \langle w^3 \rangle$ , therefore  $B_0(G) \cong \mathbb{Z}/3\mathbb{Z}$ .

We have managed to find all solvable groups  $G$  of order  $\leq 729$ , apart from the orders 512, 576 and 640, with  $B_0(G) \neq 0$ . The numbers of such groups are given in Table 1. As for the timings, it takes, for example, about seven seconds to compute  $B_0(G)$  for a given group  $G$  of order 729. We note here that the algorithm works well even for reasonably larger solvable groups. For example, the free 2-generator Burnside group  $B(2, 4)$  of exponent 4 has order  $2^{12}$ , and our algorithm returns  $B_0(B(2, 4)) \cong \mathbb{Z}/2\mathbb{Z}$ .

$n$	# of groups of order $n$	# of $G$ with $B_0(G) \neq 0$
64	267	9
128	2328	230
192	1543	54
243	67	3
256	56092	5953
320	1640	54
384	20169	1820
448	1396	54
486	261	3
704	1387	54
729	504	85

TABLE 1. Numbers of groups  $G$  with  $B_0(G) \neq 0$ .

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